COMBINATORIAL RULES FOR THREE BASES OF POLYNOMIALS

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1. INTRODUCTION

We present combinatorial rules (one theorem and two conjectures) concerning three bases of $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$.

Consider a basic question (studied for example in [L13+]):

> How does one lift properties of the ring Λ of symmetric functions (and its Schur basis) to the entirety of Pol?

The bases below lift the Schur polynomials. However, one wishes to analogize the relationship in Λ between rules for Schur polynomials and Littlewood-Richardson rules. For these bases, no rule has yet provided a parallel, explaining a desire for alternative forms.

First, we prove a "splitting" rule for the basis of *key polynomials* $\{\kappa_\alpha | \alpha \in \mathbb{Z}_{\geq 0}^{\infty}\}$, thereby establishing a new positivity theorem about them. This family was introduced by [D74] and first studied combinatorially in [LS89, LS90]. Combinatorial rules for their monomial expansion are known, see, e.g., [LS89, LS90, RS95, HHL09]. Our rule refines [RS95, Theorem 5(1)] and is compatible with the splitting rule [BKTY04, Corollary 3] for the basis of *Schubert polynomials* $\{\mathfrak{S}_w | w \in S_\infty\}$.

Second, we investigate a basis $\{\Omega_\alpha | \alpha \in \mathbb{Z}_{\geq 0}^{\infty}\}$ defined by [L01] that deforms the key basis. By extending the *Kohnert moves* of [K90] we conjecturally give the first combinatorial rule for the Ω-polynomials.

Third, in [K90], the Kohnert moves were used to conjecture the first combinatorial rule for Schubert polynomials (a proof was later presented in [W03]). Similarly, we use the extended Kohnert moves to give a conjecture for the basis of *Grothendieck polynomials* $\{\mathfrak{G}_w | w \in S_\infty\}$ [LS82]. This rule appears significantly different than earlier (proved) rules, such as those in [FK94, L01, BKTY05, LRS06].

1.1. **Splitting key polynomials.**

Let S_∞ be the group of permutations of $\mathbb{N}$ with finitely many non-fixed points. This acts on Pol by permuting the variables. Let s_i be the simple transposition interchanging x_i and x_{i+1}. The **divided difference operator** acts on Pol by

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$

Define the **Demazure operator** by setting

$$\pi_i(f) = \partial_i(x_i \cdot f), \text{ for } f \in \mathsf{Pol}.$$

For $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0}^{\infty}$, the **key polynomial** κ_α is

$$\kappa_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots, \text{ if } \alpha \text{ is weakly decreasing.}$$

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Otherwise,

$$\kappa_\alpha = \pi_i(\kappa_{\widehat{\alpha}}) \text{ where } \widehat{\alpha} = (\ldots, \alpha_{i+1}, \alpha_i, \ldots) \text{ and } \alpha_{i+1} > \alpha_i.$$

Since the leading term of κ_α is $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, the key polynomials form a $\mathbb{Z}$-basis of Pol.

The key polynomials lift the Schur polynomials: when

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t, 0, 0, 0, \ldots), \text{ where } \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_t, \text{ then} \tag{1}$$

$$\kappa_\alpha = s_{(\alpha_t, \cdots, \alpha_2, \alpha_1)}(x_1, \ldots, x_t). \tag{2}$$

A **descent** of α is an index i such that $\alpha_i \geq \alpha_{i+1}$; a **strict descent** is an index i such that $\alpha_i > \alpha_{i+1}$. Fix descents $d_1 < d_2 < \ldots < d_k$ of α containing all strict descents of α. Since π_i symmetrizes $\{x_i, x_{i+1}\}$, κ_α is separately symmetric in each collection:

$$X_1 = \{x_1, x_2, \ldots, x_{d_1}\}, X_2 = \{x_{d_1+1}, x_{d_1+2}, \ldots, x_{d_2}\}, \ldots, X_k = \{x_{d_{k-1}+1}, x_{d_{k-1}+2}, \ldots, x_{d_k}\}.$$

(The variables $x_{d_k+1}, x_{d_k+2}, \cdots$ do not appear in κ_α.) Therefore, uniquely:

$$\kappa_\alpha(X) = \sum_{\lambda^1, \ldots, \lambda^k} \mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \, s_{\lambda^1}(X_1) \cdots s_{\lambda^k}(X_k), \tag{3}$$

where each λ^i is a partition. *A priori* one only knows $\mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \in \mathbb{Z}$.

Given $\alpha \in \mathbb{Z}^\infty_{\geq 0}$, there is a unique $w[\alpha] \in S_\infty$ such that $\mathtt{code}(w[\alpha]) = \alpha$ (see, e.g., [M01, Proposition 2.1.2]). Here $\mathtt{code}(w[\alpha]) \in \mathbb{Z}^\infty_{\geq 0}$ counts the number of boxes in columns of $\mathtt{Rothe}(w[\alpha])$. We will need a special tableau coming from [S84, Section 4]:

The tableau $T[\alpha]$: Given $w[\alpha]$, $i_1 < i_2 < \ldots < i_a$ in the first column of $T[\alpha]$ are given by having i_j be the largest descent position smaller than i_{j+1} in the permutation $ws_{i_a}s_{i_{a-1}} \cdots s_{i_{j+1}}$. The next column of $T[\alpha]$ is similarly determined, starting from $ws_{i_a} \cdots s_{i_1}$, etc.

An **increasing tableau** T of shape λ is a filling with strictly increasing rows and columns. (In fact, $T[\alpha]$ is an increasing tableau.) Let $\mathtt{row}(T)$ be the reading word of T, obtained by reading the entries of T along rows, from right to left, and from top to bottom. Let $\min(T)$ be the smallest label in T. Finally, given a reduced word $\mathbf{a} = a_1 a_2 \ldots a_m$, let $\mathtt{EGLS}(\mathbf{a})$ be the output of the *Edelman-Greene correspondence* (see Section 2.1).

The following result shows $\mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k} \in \mathbb{Z}_{\geq 0}$. It is analogous to one on Schubert polynomials [BKTY04, Corollary 3] (which our proof uses).

Theorem 1.1. *The number $\mathcal{E}^\alpha_{\lambda^1, \ldots, \lambda^k}$ counts sequences of increasing tableaux $(T_1, T_2, \ldots, T_k)$ where*

- *T_i is of shape λ^i;*
- *$\min T_1 > 0, \min T_2 > d_1, \min T_3 > d_2, \ldots, \min T_k > d_{k-1}$; and*
- *$\mathtt{row}(T_1) \cdot \mathtt{row}(T_2) \cdots \mathtt{row}(T_k)$ is a reduced word of $w[\alpha]$ such that $\mathtt{EGLS}(\mathtt{row}(T_1) \cdot \mathtt{row}(T_2) \cdots \mathtt{row}(T_k)) = T[\alpha]$.*

When $d_j = j$ for all $j \geq 1$, Theorem 1.1 specializes to an instance of the monomial expansion formula [RS95, Theorem 5(1)] for κ_α (restated as Theorem 2.5 below). Also, when (1) holds, $k = 1$, $d_1 = t$ and thus Theorem 1.1 gives (2).

Example 1.2. The (strict) descents of $\alpha = (1, 3, 0, 2, 2, 1)$ are $d_1 = 2, d_2 = 5$, and

$$\kappa_{1,3,0,2,2,1} = s_{3,2}(x_1, x_2)s_{2,1,1}(x_3, x_4, x_5) + s_{3,2}(x_1, x_2)s_{2,1}(x_3, x_4, x_5)s_1(x_6) + s_{3,1}(x_1, x_2)s_{2,2}(x_3, x_4, x_5)s_1(x_6) + s_{3,1}(x_1, x_2)s_{2,2,1}(x_3, x_4, x_5).$$

exhibits the claimed non-negativity of Theorem 1.1.

Also, $w[\alpha] = 2516743$ (one line notation) and $T[\alpha] = \begin{array}{ccc} 1&3&4\\ 2&5& \\ 4&6& \\ 5&& \\ 6&& \end{array}$. Thus, $\mathcal{E}^{(1,3,0,2,2,1)}_{(3,2),(2,1,1),\emptyset} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,2),(2,1),(1)} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,1),(2,2),(1)} = \mathcal{E}^{(1,3,0,2,2,1)}_{(3,1),(2,2,1),\emptyset} = 1$ are respectively witnessed by

$$\left(\begin{array}{ccc} 1&3&4\\ 2&5& \end{array}, \begin{array}{cc} 4&6\\ 5& \\ 6& \end{array}, \emptyset\right), \left(\begin{array}{ccc} 1&3&4\\ 2&5& \end{array}, \begin{array}{cc} 4&6\\ 5& \end{array}, \begin{array}{c} 6 \end{array}\right), \left(\begin{array}{ccc} 1&3&4\\ 2&& \end{array}, \begin{array}{cc} 4&5\\ 5&6 \end{array}, \begin{array}{c} 6 \end{array}\right), \text{ and } \left(\begin{array}{ccc} 1&3&4\\ 2&& \end{array}, \begin{array}{cc} 4&5\\ 5&6\\ 6& \end{array}, \emptyset\right).$$

For example, for the leftmost sequence, $\mathtt{EGLS}(43152 \cdot 6456 \cdot \emptyset) = T[\alpha]$ holds. $\square$

1.2. **The Ω polynomials.** A. Lascoux [L01] defines Ω_α for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}^{\infty}_{\geq 0}$ by replacing π_i in the definition of the key polynomials with the operator defined by

$$\widetilde{\pi}_i(f) = \partial_i(x_i(1 - x_{i+1})f).$$

The initial condition is $\Omega_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots (= \kappa_\alpha)$, if α is weakly decreasing.

The **skyline diagram** is $\mathtt{Skyline}(\alpha) = \{(i, y) : 1 \leq y \leq \alpha_i\} \subset \mathbb{N}^2$. Graphically, it is a collection of columns α_i high. For instance,

$$\mathtt{Skyline}(1,3,0,2,2,1) = \begin{pmatrix} . & + & . & . & . & . \\ . & + & . & + & + & . \\ + & + & . & + & + & + \end{pmatrix}$$

Beginning with $\mathtt{Skyline}(\alpha)$, **Kohnert's rule** [K90] generates diagrams D by sequentially moving any $+$ at the top of its column to the rightmost open position in its row and to its left. (The result of such a move need not be the skyline of any $\gamma \in \mathbb{Z}^{\infty}_{\geq 0}$.) Let $x^D = \prod_i x_i^{d_i}$ be the column weight where d_i is the number of $+$'s in column i of D. If the same D results from a different sequence of moves, it only counts once. Kohnert's theorem states $\kappa_\alpha = \sum x^D$, where the sum is over all such D. Extending this, we introduce:

The K-Kohnert rule: Each $+$ either moves as in Kohnert's rule, or stays in place *and* moves. In the latter case, mark the original position with a "g". The g's are unmovable, but a given $+$ treats g the same as other $+$'s when deciding if it can move, and to where. Diagrams with the same occupied positions but different arrangements of $+$'s and g's are counted separately.

Example 1.3. Below, we give all K-Kohnert moves one step from D:

$$D = \begin{pmatrix} + & . & g & + & . \\ . & + & + & + & + \end{pmatrix} \mapsto \begin{pmatrix} + & . & g & + & . \\ + & . & + & + & + \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & g & + & + & + \end{pmatrix}, \begin{pmatrix} + & + & g & . & . \\ . & + & + & + & + \end{pmatrix},$$

$$\begin{pmatrix} + & + & g & g & . \\ . & + & + & + & + \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & + & + & + & . \end{pmatrix}, \begin{pmatrix} + & . & g & + & . \\ + & + & + & + & g \end{pmatrix}.$$

Let

$$J_\alpha^{(\beta)} = \sum \beta^{(\#g\text{'s appearing in } D)} x^D.$$

Conjecture 1.4. $J_\alpha^{(-1)} = \Omega_\alpha$.

Conjecture 1.4 has been checked by computer, for a wide range of cases up to α being of size 12, leaving us convinced. Clearly, $J_\alpha^{(0)} = \kappa_\alpha$, by Kohnert's theorem.

Example 1.5. Let $\alpha = (1,0,2)$. Then the diagrams contributing to $J_{(1,0,2)}$ are:

$$\texttt{Skyline}(1,0,2) = \begin{pmatrix} \cdot & \cdot & + \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & + & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & + & \cdot \end{pmatrix};$$

$$\begin{pmatrix} + & g & \cdot \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & g & \cdot \\ + & + & \cdot \end{pmatrix}, \begin{pmatrix} + & \cdot & \cdot \\ + & + & g \end{pmatrix}, \begin{pmatrix} \cdot & + & \cdot \\ + & + & g \end{pmatrix}, \begin{pmatrix} \cdot & + & g \\ + & \cdot & + \end{pmatrix}, \begin{pmatrix} + & \cdot & g \\ + & \cdot & + \end{pmatrix}; \begin{pmatrix} + & g & \cdot \\ + & + & g \end{pmatrix}; \begin{pmatrix} + & g & g \\ + & \cdot & + \end{pmatrix}.$$

Thus

$$\begin{aligned} J_{(1,0,2)} &= (x_1x_3^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2) \\ &\quad - (x_1^2x_2x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_3^2) + (x_1^2x_2^2x_3 + x_1^2x_2x_3^2). \end{aligned}$$

The lowest degree homogeneous component of Ω_α is κ_α. Hence any $f \in \mathsf{Pol}$ is a possibly *infinite* linear combination of the Ω_α's. Finiteness is asserted in [L13+, Chapter 5]. We show in Section 4.2 that the J_α's also form a (finite) basis.

1.3. **Grothendieck polynomials.** The **Grothendieck polynomial** [LS82] is defined using the **isobaric divided difference operator** whose action on $f \in \mathsf{Pol}$ is given by:

$$\pi_i(f) = \partial_i((1 - x_{i+1})f).$$

Declare $\mathfrak{G}_{w_0}(X) = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ where w_0 is the long element in S_n. Set $\mathfrak{G}_w(X) = \pi_i(\mathfrak{G}_{ws_i})$ if i is an ascent of w. The Grothendieck polynomials are known to lift $\{s_\lambda\}$ to Pol.

One has $\mathfrak{G}_w = \mathfrak{S}_w +$ (higher degree terms). We now state the A. Kohnert's conjecture [K90] for $\mathfrak{G}_w$. The **Rothe diagram** is $\texttt{Rothe}(w) = \{(x,y) | y < w(x) \text{ and } x < w^{-1}(y)\} \subset [n] \times [n]$ (indexed so that the southwest corner is labeled $(1,1)$). Starting with $\texttt{Rothe}(w)$, the Kohnert's rule generates diagrams D by applying the same rules as described for his rule for κ_α. Then $\mathfrak{S}_w = \sum x^D$; the sum is over all such D.

Analogously, we define

$$K_w^{(\beta)} = \sum_D \beta^{(\#g\text{'s appearing in } D)} \mathbf{x}^D$$

where the sum is over all diagrams D generated by the K-Kohnert rule. For example, if $w = 3142$ the diagrams contributing to $K_w^{(\beta)}$ are

$$\texttt{Rothe}(3142) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & + & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & + & g & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}.$$

and hence correspondingly, $K_{3142}^{(-1)} = (x_1^2x_3 + x_1^2x_2) - (x_1^2x_2x_3)$.

Conjecture 1.6. $K_w^{(-1)} = \mathfrak{G}_w$.

Note, $K_w^{(0)} = \mathfrak{S}_w$ is precisely Kohnert's conjecture. Conjecture 1.6 has been checked by computer for $n \leq 7$, and extensively for larger n. While Kohnert's rule for $\mathfrak{S}_w$ is handy, it remains mysterious, even after [W03]. Conjectures 1.4 and 1.6 return to Kohnert's conjecture (albeit with a parameter β).

2. Proof of Theorem 1.1

2.1. **Reduced word combinatorics.** Given $w \in S_n$, let

$$\mathbf{a} = (a_1, a_2, \ldots a_{\ell(w)}) \text{ and } \mathbf{i} = (i_1, i_2, \ldots, i_{\ell(w)}).$$

In connection to [BJS93], we say the pair $(\mathbf{a}, \mathbf{i})$ is a **stable compatible pair for** w if $s_{a_1} \cdots s_{a_{\ell(w)}}$ is a reduced word for w and the following two conditions on $\mathbf{i}$ hold:

(cs.1) $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{\ell(w)} < n$;
(cs.2) $a_j < a_{j+1} \implies i_j < i_{j+1}$.

We will identify w with $\mathbf{a}$ and the associated reduced word.

The **Edelman-Greene correspondence** [EG87] (the same basic construction is used in [LS82]) is a bijection

$$\texttt{EGLS} : (\mathbf{a}, \mathbf{i}) \mapsto (T, U)$$

where

- T is an increasing tableau such that $\texttt{row}(T)$ is a reduced word for $\mathbf{a}$; and
- U is a semistandard tableau whose multiset of labels is precisely those in $\mathbf{i}$, and which has the same shape as T.

$\texttt{EGLS}$ (column) insertion: Initially insert a_j into the leftmost column (of what will be T). If there are no labels strictly larger than a_j, we place a_j at the bottom of that column. If $a_j + t$ for $t > 2$ appears, we bump this $a_j + t$ to the next column to the right, replacing it with a_j. The same holds if $a_j + 1$ appears but not a_j. Finally, if both $a_j + 1$ and a_j already appear, we insert $a_j + 1$ into the next column to the right. Since $\mathbf{a}$ is assumed to be reduced, the above enumerates all possibilities. Finally at step j a new box is created at a corner; in what will be U we place i_j .

Mildly abusing terminology, let $\texttt{EGLS}(\mathbf{a}) = T$.

2.2. **Formulas for Schubert polynomials.** A stable compatible pair $(\mathbf{a}, \mathbf{i})$ is a **compatible pair for** w if in addition to (cs.1) and (cs.2) the following holds:

(cs.3) $i_j \leq a_j$.

Let $\texttt{Compatible}(w)$ be the set of compatible sequences for w. A rule of [BJS93] states:

$$\mathfrak{S}_w(X) = \sum_{(\mathbf{a},\mathbf{i}) \in \texttt{Compatible}(w)} \mathbf{x}^{\mathbf{i}}. \tag{4}$$

A **descent** of w is an index j such that $w(j) > w(j+1)$. Let $\texttt{Descents}(w)$ be the set of descents of w. The following is [BKTY04, Corollary 3]:

Theorem 2.1. *Let $w \in S_n$ and suppose $\texttt{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\}$. Then*

$$\mathfrak{S}_w(X) = \sum_{\lambda^1, \ldots, \lambda^k} c^w_{\lambda^1, \ldots, \lambda^k} s_{\lambda^1}(X_1) \cdots s_{\lambda^k}(X_k) \tag{5}$$

where $c^w_{\lambda^1, \ldots, \lambda^k}$ counts the number of tuples of increasing tableaux $(T_1, \ldots, T_k)$ where

(i) *T_i has shape λ^i;*
(ii) *$\min T_1 > 0, \min T_2 > d_1, \ldots, \min T_k > d_{k-1}$; and*
(iii) *$\texttt{row}(T_1) \cdots \texttt{row}(T_k)$ is a reduced word of w.*

Assume for the remainder of the proof that

$$\texttt{Descents}(w) \subseteq \{d_1 < d_2 < \ldots < d_k\}. \tag{6}$$

Let

$$\texttt{Tuples}(w) = \{[(T_1, U_1), (T_2, U_2), \ldots, (T_k, U_k)]\}$$

where the T_i's satisfy (i), (ii) and (iii) from Theorem 2.1, and each U_i is a semistandard tableau of shape λ^i using the labels $d_{i-1}+1, d_{i-1}+2, \ldots, d_i$ ($d_0 = 0$).

2.3. **"Splitting" the EGLS correspondence.** Assuming (6) we define:

$$\Phi : \texttt{Compatible}(w) \to \texttt{Tuples}(w).$$

Description of Φ (using EGLS): Uniquely split $(\mathbf{a}, \mathbf{i}) \in \texttt{Compatible}$ as follows

$$\left((\mathbf{a}^{(1)}, \mathbf{i}^{(1)}), (\mathbf{a}^{(2)}, \mathbf{i}^{(2)}), \cdots, (\mathbf{a}^{(k)}, \mathbf{i}^{(k)})\right) \tag{7}$$

where

- $\mathbf{a} = \mathbf{a}^{(1)} \cdots \mathbf{a}^{(k)}$ and $\mathbf{i} = \mathbf{i}^{(1)} \cdots \mathbf{i}^{(k)}$ ("$\cdots$" means concatenation); and
- the entries of $i^{(j)}$ are contained in the set $\{d_{j-1}+1, d_{j-1}+2, \cdots, d_j\}$.

Now define

$$\Phi((\mathbf{a}, \mathbf{i})) := \left(\texttt{EGLS}(\mathbf{a}^{(1)}, \mathbf{i}^{(1)}), \cdots, \texttt{EGLS}(\mathbf{a}^{(k)}, \mathbf{i}^{(k)})\right).$$

Proposition 2.2. *The map $\Phi : \texttt{Compatible}(w) \to \texttt{Tuples}(w)$ is well-defined and a bijection.*

Proof. Φ is well-defined: The condition (i) is just says T_j and U_j have the same shape, which is true by EGLS's description. For (ii), the splitting says each label in $\mathbf{i}^{(j)}$ is strictly bigger than d_{j-1}. Now by (cs.3), each label in $\mathbf{a}^{(j)}$ is strictly bigger than d_{j-1} as well. By EGLS's definition, the set of labels appearing in T_j is the same as that of $\mathbf{a}^{(j)}$; hence (ii) holds. Lastly, $\texttt{row}(T_j)$ is a reduced word for $a^{(j)}$. Then (iii) is clear.

Φ is a bijection: Since EGLS is a bijective correspondence, clearly Φ is an injection. Consider the weight function on $\texttt{Compatible}(w)$ that assigns $(\mathbf{a}, \mathbf{i})$ weight $\mathbf{x}^{\mathbf{i}}$ and assigns $[(T_1, U_1), \ldots, (T_k, U_k)]$ the weight $\mathbf{x}^{U_1} \cdots \mathbf{x}^{U_k}$, where $\mathbf{x}^{U_i}$ is the usual monomial associated to the tableau U_i. Then clearly Φ is a weight-preserving map (since EGLS is similarly weight-preserving). Hence the surjectivity of Φ holds by (4) and Theorem 2.1. ☐

See [L04, Section 5] for a proof of Theorem 2.1 which is close to the study of the split EGLS correspondence (the argument constructs certain crystal operators).

2.4. **The tableau $T[\alpha]$.** Recall $w[\alpha] \in S_\infty$ satisfies $\texttt{code}(w[\alpha]) = \alpha$. Let $\prec$ be the pure reverse lexicographic total ordering on monomials. The Schubert polynomial $\mathfrak{S}_{w[\alpha]}$ has leading term $\mathbf{x}^\alpha$ (with respect to $\prec$). The same is true of κ_α (see [RS95, Corollary 7]) so

$$\mathfrak{S}_{w[\alpha]} = \kappa_\alpha + \text{linear combination of other key polynomials.} \tag{8}$$

Given an increasing tableau U, the **nil left key** $K_-^0(U)$ is defined by [LS89] (cf. [RS95, p.111–114]). Let $\texttt{sort}(\alpha)$ be the partition obtained by rearranging α into weakly decreasing order. Also let $\texttt{content}(T)$ the usual content vector of a semistandard tableau T. This is a result of A. Lascoux-M.-P. Schützenberger (cf. [RS95, Theorem 4]):

Theorem 2.3.
$$\mathfrak{S}_w(X) = \sum \kappa_{\texttt{content}(K_-^0(U))}$$
where the sum is over all increasing tableaux U of shape $\texttt{sort}(\alpha)$ *with* $\texttt{row}(U) = w$.

Thus, by (8) combined with Theorem 2.3 there exists a unique increasing tableau $U[\alpha]$ of shape $\texttt{sort}(\alpha)$ with $\texttt{row}(U[\alpha]) = w[\alpha]$ and such that $\alpha = \texttt{content}(K_-^0(U[\alpha]))$.

Let $F_w = \lim_{k\to\infty} \mathfrak{S}_{1^k \times w}$ be the **stable Schubert polynomial** associated to w. This is a symmetric polynomial in infinitely many variables. So therefore one has an expansion

$$F_w = \sum_{\lambda} a_{w,\lambda} s_\lambda, \tag{9}$$

where the $a_{w,\lambda} \in \mathbb{Z}_{\geq 0}$ are counted by increasing tableaux A of shape λ with $\texttt{row}(A) = w$.

In [S84, Theorem 4.1], it is shown $a_{w,\mu(w)'} = 1$ for a certain explicitly described "maximal" $\mu'(w)$. Moreover a simple description of the witnessing tableau $A[\alpha]$ is given. Straightforwardly, $\mu'(w[\alpha]) = \texttt{sort}(\alpha)$. Then $T[\alpha]$ is precisely the witnessing tableau $A[\alpha]$ for $a_{w[\alpha],\lambda(w[\alpha])}$ (after accounting for the fact that [S84]'s conventions use $F_{w[\alpha]}$ for what we call $F_{w[\alpha]^{-1}}$). We leave the details to the reader.

Finally, the expansion of Theorem 2.3 refines (9); see, e.g., [RS95]. Hence, $T[\alpha] = A[\alpha] = U[\alpha]$. So, $T[\alpha]$ is an increasing tableau of shape $\texttt{sort}[\alpha]$ with $\texttt{row}(T[\alpha]) = w[\alpha]$ and $\texttt{content}(K_-(T[\alpha])) = \alpha$.

2.5. **Conclusion of the proof of Theorem 1.1:** From the definition of $\texttt{Rothe}(w[\alpha])$:

Lemma 2.4. *The descents of $w[\alpha]$ are contained in the set of descents $d_1 < d_2 < \ldots < d_k$ of α.*

Thus,

$$\mathfrak{S}_{w[\alpha]}(X) = \sum_{(\mathbf{a},\mathbf{i})} \mathbf{x}^{\mathbf{i}} = \sum_{\lambda^1,\ldots,\lambda^k} c^{w[\alpha]}_{\lambda^1,\ldots,\lambda^k} s_{\lambda^1}(X_1)\cdots s_{\lambda^k}(X_k). \tag{10}$$

We recall a formula [RS95, Theorem 5]:

Theorem 2.5. *Fix an increasing tableau T with* $\texttt{content}(K_-^0(T)) = \alpha$. *Then*
$$\kappa_\alpha = \sum_{(\mathbf{a},\mathbf{i})} \mathbf{x}^{\mathbf{i}}$$
where the sum is over compatible sequences $(\mathbf{a},\mathbf{i})$ satisfying (cs.1), (cs.2), (cs.3) and $\texttt{EGLS}(\mathbf{a}) = T$.

Two reduced words $\mathbf{a}$ and $\mathbf{a}'$ for the same permutation are in the same **Coxeter-Knuth class** if $\texttt{EGLS}(\mathbf{a}) = \texttt{EGLS}(\mathbf{a}') = T$. This T **represents** the class. This equivalence relation $\sim$ on reduced words is defined by the symmetric and transitive closure of the relations:

$$\begin{aligned} \mathbf{A}i(i+1)i\mathbf{B} &\sim \mathbf{A}(i+1)i(i+1)\mathbf{B} \\ \mathbf{A}acb\mathbf{B} &\sim \mathbf{A}cab\mathbf{B} \\ \mathbf{A}bac\mathbf{B} &\sim \mathbf{A}bca\mathbf{B} \end{aligned} \tag{11}$$

where $a < b < c$. In particular, it is true that $\mathbf{a} \sim \texttt{row}(\texttt{EGLS}(\mathbf{a}))$.

Restrict Φ to those $(\mathbf{a},\mathbf{i}) \in \texttt{Compatible}(w[\alpha])$ such that $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$. Consider $\Phi(\mathbf{a},\mathbf{i}) = [(T_1,U_1),\ldots,(T_k,U_k)]$. Since $\texttt{EGLS}(\mathbf{a}^{(i)}) \sim \texttt{row}(T_i)$, by (11) we see

$$\texttt{row}(T_1)\cdots\texttt{row}(T_k) \sim \mathbf{a}^{(1)}\cdots\mathbf{a}^{(k)} = \mathbf{a}. \tag{12}$$

However, since we have assumed $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$, therefore:

$$\texttt{EGLS}(\texttt{row}(T_1)\cdots\texttt{row}(T_k)) = T[\alpha], \tag{13}$$

The other two requirements on $(T_1, \ldots, T_k)$ hold since Φ is well-defined.

Conversely, suppose $[(T_1, U_1), \ldots, (T_k, U_k)]$ has $(T_1, \ldots, T_k)$ satisfying Theorem 1.1's conditions. Since Φ is a bijection, $\Phi^{-1}([(T_1, U_1), \ldots, (T_k, U_k)]) = (\mathbf{a}, \mathbf{i}) \in \texttt{Compatible}(w[\alpha])$. Also, by (12), $\mathbf{a} \sim \texttt{row}(T_1)\cdots\texttt{row}(T_k)$. Now, we assumed (13) holds. Hence, $\texttt{EGLS}(\mathbf{a}) = T[\alpha]$ as desired. This completes the proof of the Theorem 1.1. $\square$

3. Additional remarks

3.1. **Comments on Theorem 1.1.** Since κ_α specialize non-symmetric Macdonald polynomials (see, e.g., [HHL09, Section 5.3]), can one extend Theorem 1.1 in that direction?

Theorem 1.1 implies that the key module of [RS95, Section 5] should have an action of $GL(d_1) \times GL(d_2 - d_1) \times \cdots \times GL(d_k - d_{k-1})$ such that the character is κ_α.

V. Reiner suggests a variation of Theorem 1.1 using the plactic theory. The derivation should be similar, using formulas from [RS94]. However we are missing the analogue of [BKTY04, Corollary 4]; cf. [KMS06, Sections 7, 8]. Theorem 1.1 naturally generalizes to Grothendieck polynomials, using [BKTY05, BKSTY08]; details may appear elsewhere.

3.2. **J_α's form a (finite) basis of Pol.** Clearly, $J_\alpha(X) = \mathbf{x}^\alpha + \sum_{\beta \prec \alpha} c_\beta \mathbf{x}^\beta$. One decomposes $f \in \mathsf{Pol}$ into a possibly infinite sum of J_α's:

$$f = \sum_\alpha g_\alpha J_\alpha \tag{14}$$

That is, find the $\prec$ largest monomial $\mathbf{x}^{\theta_0}$ appearing in $f^{(0)} := f$ (say with coefficient c_{θ_0}) and let $f^{(1)} := f - c_{\theta_0} \cdot J_{\theta_t}$. Thus $f^{(1)}$ only contains monomials strictly smaller in the $\prec$ ordering. Now repeat, defining $f^{(t+1)} := f^{(t)} - c_{\theta_t} J_{\theta_t}$ where $\mathbf{x}^{\theta_t}$ is the $\prec$-largest monomial appearing in $f^{(t)}$ etc. Since J_α is not homogeneous, each step t potentially introduces $\prec$-smaller monomials but of higher degree. However, we claim:

Proposition 3.1. *The expansion (14) is finite.*

Proof. By the K-Kohnert rule, each β that appears in J_α is contained in the smallest rectangle R that contains α. So the above procedure only involves the finitely many diagrams contained in R for one of the finitely many initial $\alpha \in \mathbb{Z}_{\geq 0}^\infty$ such that $\mathbf{x}^\alpha$ is in f. $\square$

3.3. **More on the interplay of Grothendieck and the Ω polynomials.** M. Shimozono has suggested that the expansion of $\mathfrak{G}_w$ into Ω_α should alternate in sign, by degree. An explicit rule exhibiting this has been conjectured by V. Reiner and the second author.

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